ADDENDUM TO *INVARIANT MEANS AND CONES WITH VECTOR INTERIORS*

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1. **Introduction.** In the preceding paper [4] it is proved that the monotone extension property (Hahn-Banach extension property) of a pair $[\bar{G}, V]$, where $\bar{G}$ is a semi-group and V is a boundedly complete vector lattice whose positive cone is sharp and has a vector interior point, is equivalent to the property that there exists an invariant mean definable on the space of bounded real valued functions on $\bar{G}$. It is the purpose of this paper to remove the interior point restriction on V.

The notations and definitions are the same as in [4] except the following slight modification. A semi-group $\bar{G}$ has the *monotone* (*Hahn-Banach*) *extension property* if and only if $[\bar{G}, V]$ has the monotone (Hahn-Banach) extension property [2] for every boundedly complete vector lattice V whose positive cone is sharp.

2. Theorem. *If $\bar{G}$ is a semi-group the following statements are equivalent:*
(1) *$\bar{G}$ has the monotone extension property.*
(2) *$\bar{G}$ has the Hahn-Banach extension property.*
(3) *$\bar{G}$ has an invariant mean.*

The equivalence of (1) and (2) has been proved in [2], that (1) and (2) imply (3) has been proved in [3]. One needs, therefore, only consider the proof that (3) implies (1).

Case A. V has reproducing positive cone K. Consider a collection $[Y, X, C, f, G]$ associated with V and K as in [4, Definition 1], where G is a representation of $\bar{G}$ on Y, and $\bar{G}$ has an invariant mean. Let Γ denote the set of convex combinations of elements $g \in G$ and the identity, and let $\sigma, \tau, \cdots$ denote the elements of Γ. It is sufficient to consider the case that Y is spanned by X and $\{\sigma y_0 | \sigma \in \Gamma\}$, where y_0 is a fixed element in Y not in X. Since $(y_0 + X) \cap C \neq \emptyset$, it may be assumed that $y_0 > 0$.

Let $A = \{x \in X | x \leqq y_0\}$, $B = \{x \in X | x \geqq y_0\}$. The sets A and B are not empty since $(y_0 + X) \cap C \neq \emptyset$ and $(-y_0 + X) \cap C \neq \emptyset$. Consider a fixed $b \in B$. Define $W = \{w \in V | w \geqq f(b)\}$. If $w \in W$, then $w \geqq f(b) \geqq 0$. For each $w \in W$ define

$$V_w = \{v \in V \,|\, tw \geqq \sup (v, -v) \text{ for some } t > 0\},$$

$$X_w = \{x \in X \,|\, f(x) \in V_w\},$$

$$Y_w = \text{the subspace spanned by } X_w \text{ and } \{\sigma y_0 \,|\, \sigma \in \Gamma\}.$$

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Then

LEMMA. (1) *V_w is a linear subspace of V.*

(2) *X_w is a linear subspace of X.*

(3) *The vector w is a vector interior point of the positive cone $K_w = K \cap V_w$ in V_w and K_w is sharp.*

(4) *V_w is a boundedly complete vector lattice.*

(5) *For each $x \in X$, there exists $w \in W$ such that $x \in X_w$, hence $\bigcup_{w \in W} X_w = X$, $\bigcup_{w \in W} Y_w = Y$.*

(6) *$gX_w \subset X_w$, $gC_w \subset C_w (C_w = C \cap Y_w)$, $w \in W$, $g \in G$, and $(y + X_w) \cap C_w \neq \varnothing$, $y \in Y_w$.*

(7) *If $x = s\sigma y_0 - t\tau y_0 \in X$, where $s, t \geqq 0$, then $x \in X_w$ for all $w \in W$.*

(8) *The system W forms a directed system under $\geqq$. If $w \geqq w'$, then $X_w \supseteq X_{w'}$, $Y_w \supseteq Y_{w'}$, $C_w \supseteq C_{w'}$, $V_w \supseteq V_{w'}$, $K_w \supseteq K_{w'}$.*

(9) *For every V_w, K_w and associated $[Y_w, X_w, C_w, f_w, G]$, where f_w is the restriction of f to X_w, there exists a monotone invariant extension F_w: $Y_w \rightarrow V_w$ of f_w.*

Proof of lemma. Ad (1). If $v, v' \in V_w$, then there exist $t, t' > 0$ such that $tw > \sup(v, -v)$ and $t'w > \sup(v', -v')$. Then $(t+t')w > \sup(v+v', -v-v')$. Therefore $v+v' \in V_w$. Clearly, $-v \in V_w$ and $tv \in V_w$ for all $t > 0$.

Ad (2). This follows from (1) and the linearity of f.

Ad (3). That w is a vector interior point of K_w follows easily from the facts that $w > 0$ and $tw \pm v \geqq 0$ for some $t > 0$. K_w is sharp because K is sharp.

Ad (4). Let S be a subset of V_w bounded from above by v_1 in V_w. Then S has a least upper bound v in V. Take any vector v_2 in S. There exist $t_1, t_2 > 0$ such that $t_1 w > \sup(v_1, -v_1)$ and $t_2 w > \sup(v_2, -v_2)$. Then $\max(t_1, t_2)w > \sup(v_1, -v_1, v_2, -v_2) \geqq \sup(v, -v)$, since $v_1 \geqq v$ and $-v_2 \geqq -v$. Hence $v \in V_w$.

Ad (5). For each $x \in X$, the vector $w = \sup(f(b), f(x), -f(x))$ is in W and $x \in X_w$.

Ad (6). X_w and C_w are invariant under G because f and C are invariant under G. That every translate of X_w intersects C_w follows from the facts $y_0 \geqq 0$, $b - y_0 \geqq 0$ and $b \in X_w$.

Ad (7). If $x = (s\sigma - t\tau)y_0 \in X$ then $x \leqq s\sigma b$ and $-x \leqq t\tau b$. Hence

$$\sup(f(x), -f(x)) \leqq \max(s, t) f(b),$$

consequently, $x \in X_w$ for every $w \in W$.

Ad (8). This is clear.

Ad (9). Statement (9) follows from (1)–(8) and [4, Theorem 1].

Let $u_0 = \inf_{w' \in W} \sup_{w \geqq w'} F_w(y_0)$. The vector u_0 exists since $0 \leqq F_w(y_0) \leqq f(b)$, $w \in W$. Define F: $Y \rightarrow V$ by

$$F(x + s\sigma y_0 - t\tau y_0) = f(x) + (s - t)u_0, \qquad s, t \geqq 0.$$

The function F is well defined, that is, if $x + s\sigma y_0 - t\tau y_0 = x' + s'\sigma' y_0 - t'\tau' y_0$

then $f(x)+(s-t)u_0=f(x')+(s'-t')u_0$; or equivalently, if $(s\sigma-t\tau)y_0=x\in X$ then $f(x)=(s-t)u_0$. But in this case $x\in X_w$ for every w. Therefore $f(x)=F_w(s\sigma y_0-t\tau y_0)=(s-t)F_w(y_0)$ for all $w\in W$. If $s-t=0$ then $f(x)=0=(s-t)u_0$. If $s-t\neq 0$ then $F_w(y_0)=f(x)/(s-t)$ is independent of w. Hence $u_0=F_w(y_0)$. The function F is obviously invariant under G, and by definition distributive. To prove that F is monotone, let $x+s\sigma y_0-t\tau y_0>0$. Consider $w_0\in W$, where $x\in X_{w_0}$ and thus $x\in X_w$, $w\geqq w_0$.

CASE 1. $s-t>0$. For each

$$w\geqq w'\geqq w_0,\ F_w(x+s\sigma y_0-t\tau y_0)=f(x)+(s-t)F_w(y_0)\geqq 0.$$

Since $(s-t)F_w(y_0)\geqq 0$, $f(x)+(s-t)\ \sup_{w\geqq w'} F_w(y_0)\geqq 0$, and hence

$$f(x)+(s-t)\inf_{w'\geqq w_0}\sup_{w\geqq w'}F_w(y_0)$$

$$=f(x)+(s-t)u_0=F(x+s\sigma y_0-t\tau y_0)\geqq 0.$$

CASE 2. $t-s>0$. $x>t\tau y_0-s\sigma y_0$ and as in Case 1 $f(x)\geqq(t-s)u_0$.

All that remains to complete the proof is to remove the restriction that V has reproducing cone.

CASE B. The cone of V is not necessarily reproducing. Let V_1 be the subspace of V spanned by K, $X_1=\{x\in X\mid f(x)\in V_1\}$, and let Y_1 be the subspace of Y spanned by X_1 and $\{\sigma y_0\mid\sigma\in\Gamma\}$—assuming again that Y is spanned by X and $\{\sigma y_0\mid\sigma\in\Gamma\}$ and that $y_0>0$. The subsets A, B of X are contained in X_1 for $B\geqq 0$, $B-A\geqq 0$ and $A\subset B-(B-A)$. If $x\geqq s\sigma y_0-t\tau y_0$, $x\in X$, then $x\geqq -t\tau b$ and $x=(x+t\tau b)-t\tau b\in X_1$. Consequently $C\subset Y_1$. The collection $[Y_1, X_1, C, f_1, G]$ together with V_1 and K satisfies the conditions of Case A, where f_1 is the restriction of f on X_1. Therefore there is a monotone invariant extension F_1: $Y_1\to V_1$ of f_1. Define F: $Y\to V$ by $F(x+s\sigma y_0-t\tau y_0)=f(x)+F_1(s\sigma y_0-t\tau y_0)$. It is easy to show that F is well defined and, hence, is a desired extension of f.

As an immediate corollary is a generalization of a theorem of Kreĭn and Rutman [1, Theorem 3.1].

COROLLARY. *Let Y be an ordered linear space whose positive cone C has a vector interior point y_0. Let G be a semi-group of operators on Y such that $g(C)\subset C$ and $gy_0=\lambda_g y_0$, $g\in G$, where $\lambda_g>0$. Then if G has an invariant mean, there is a positive distributive functional F on Y with $F(gy)=\lambda_g F(y)$, $g\in G$.*

Proof. Let $\bar{G}$ be the set of operators $\{\bar{g}=g/\lambda_g\mid g\in G\}$. Since $\lambda_{gg'}=\lambda_g\lambda_{g'}$, $\bar{G}$ is a homomorphic image of G and y_0 is a fixed point of $\bar{G}$. Let X be the one-dimensional subspace spanned by y_0. Since y_0 is a vector interior point $(y+X)\cap C\neq\varnothing$ for every $y\in Y$. Hence the function f: $X\to$real numbers defined by $f(y_0)=1$ can be extended to a monotone function F invariant under $\bar{G}$. Thus $F(gy)=\lambda_g F(\bar{g}y)=\lambda_g F(y)$, $y\in Y$.

That this corollary is a generalization of [1, Theorem 3.1] is assured by [1, Lemma 1.1].

References

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